

# SHAPE DERIVATIVE OF THE FIRST EIGENVALUE OF THE 1-LAPLACIAN

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**ABSTRACT.** We compute the shape derivative of the functional  $\Omega \rightarrow \lambda_{1,\Omega}$ , where  $\lambda_{1,\Omega}$  denotes the first eigenvalue of the 1-Laplacian on  $\Omega$ . As an application, we find that the ball is critical among the volume-preserving deformations.

Let  $\Omega$  be a smooth bounded domain in  $\mathbb{R}^n$ ,  $n \geq 2$ . The 1-Laplacian on  $\Omega$  is the formal operator

$$\Delta_1 u = -\operatorname{div} \left( \frac{\nabla u}{|\nabla u|} \right)$$

we get by a formal derivation of  $F(u) = \int_{\Omega} |\nabla u| dx$ , or by letting  $p \rightarrow 1$  in the definition of the  $p$ -Laplacian,  $p > 1$ . By analogy with the definition of the first eigenvalue of the  $p$ -Laplacian on  $\Omega$ , we define the first eigenvalue  $\lambda_{1,\Omega}$  of the 1-Laplacian on  $\Omega$  by the minimization problem

$$\lambda_{1,\Omega} = \inf_{\substack{u \in \dot{H}_1^1(\Omega) \\ \int_{\Omega} |u| dx = 1}} \int_{\Omega} |\nabla u| dx ,$$

where  $\dot{H}_1^1(\Omega)$  is the closure of  $C_0^\infty(\Omega)$  in the Sobolev space  $H_1^1(\Omega)$  of functions in  $L^1(\Omega)$  with one derivative in  $L^1$ .

The purpose of this paper is the study of the dependence of  $\lambda_{1,\Omega}$  under regular perturbations by diffeomorphisms of  $\Omega$ , i.e. we want to compute the first variation, the so-called shape derivative, of the functional  $\Omega \rightarrow \lambda_{1,\Omega}$ . General results about the stability of  $\lambda_{1,\Omega}$  under perturbations of  $\Omega$  have been obtained in [17]. In particular the authors of [17] found the shape derivative of  $\Omega \rightarrow \lambda_{1,\Omega}$  in the case of regular perturbations by diffeomorphisms close to homotheties. We want to extend this result to the case of a general perturbation by diffeomorphisms.

Let us recall some known facts about  $\lambda_{1,\Omega}$  (see e.g. [17, 20]). A natural space to study  $\lambda_{1,\Omega}$  is the space  $BV(\Omega)$  of functions of bounded variations (see, for instance, [2, 11, 14, 26]). By standard properties of  $BV(\Omega)$ , we can also define  $\lambda_{1,\Omega}$  by

$$\lambda_{1,\Omega} = \inf_{\substack{u \in BV(\Omega) \\ \int_{\Omega} |u| dx = 1}} \int_{\Omega} |\nabla u| + \int_{\partial\Omega} |u| dH^{n-1}, \quad (1)$$

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where  $|\nabla u|$  is the total variation of the measure  $\nabla u$ , and  $H^{n-1}$  denotes the  $(n-1)$ -dimensional Hausdorff measure. We refer to [8] for a detailed proof of this assertion. Note here that if  $u \in BV(\Omega)$  and  $\bar{u}$  is the extension of  $u$  by 0 in  $\mathbb{R}^n \setminus \bar{\Omega}$ , then  $\bar{u} \in BV(\mathbb{R}^n)$  and

$$\int_{\mathbb{R}^n} |\nabla \bar{u}| = \int_{\Omega} |\nabla u| + \int_{\partial\Omega} |u| dH^{n-1}. \quad (2)$$

By lower semicontinuity of the total variation and compactness of the embedding  $BV(\Omega) \hookrightarrow L^1(\Omega)$ , it easily follows from (2) that the infimum in (1) is attained by some nonnegative  $u \in BV(\Omega)$ . Then  $u$  is a solution of the equation  $\Delta_1 u = \lambda_{1,\Omega}$  in the sense that there exists  $\sigma \in L^\infty(\Omega, \mathbb{R}^n)$ ,  $\|\sigma\|_\infty \leq 1$ , such that

$$\begin{cases} -\operatorname{div} \sigma = \lambda_{1,\Omega}, \\ \sigma \nabla u = |\nabla u| \text{ in } \Omega, \text{ and} \\ (\sigma \vec{n})u = -u \text{ on } \partial\Omega, \end{cases} \quad (3)$$

where  $\vec{n}$  is the unit outer normal to  $\partial\Omega$ , and  $\sigma \nabla u$  is the distribution defined by integrating by parts  $\int_{\Omega} (\sigma \nabla u) v dx$  when  $v \in C_0^\infty(\Omega)$  and  $\operatorname{div} \sigma$  makes sense (see e.g. [8, 3]). We then say that  $u$  is an eigenfunction for  $\lambda_{1,\Omega}$ .

We can also express  $\lambda_{1,\Omega}$  in a more geometric way as an isoperimetric type problem. We recall that a set  $C \subset \mathbb{R}^n$  is said of finite perimeter if its characteristic function  $\chi_C$  belongs to  $BV(\mathbb{R}^n)$ . We then define the perimeter  $|\partial C|$  of  $C$  as  $\int_{\mathbb{R}^n} |\nabla \chi_C|$ . Using the coarea formula, we can rewrite (1) as

$$\lambda_{1,\Omega} = \inf_{C \subset \bar{\Omega}, \chi_C \in BV(\mathbb{R}^n)} \frac{|\partial C|}{|C|}. \quad (4)$$

We refer e.g. to [18] for a proof of this assertion. This infimum is attained by some set of finite perimeter, e.g. by a level-set of an extremal for (1), called an eigenset or also a Cheeger's set. Note that minimizers for  $\lambda_{1,\Omega}$  touch the boundary  $\partial\Omega$  since, if not, we may blow it up by a factor larger than one, which would decrease  $\lambda_{1,\Omega}$ . Uniqueness and nonuniqueness results of eigensets are in [12, 25]. Concerning regularity, possible references are [1, 10, 15, 18, 25].

To study the variations of  $\lambda_{1,\Omega}$  with respect to smooth variations of  $\Omega$ , we consider a smooth vector field  $V : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and, for small  $t \in \mathbb{R}$ , the diffeomorphisms  $T_t$  defined by

$$T_t(x) = x + tV(x), \quad (5)$$

and eventually the perturbed domains  $\Omega_t = T_t(\Omega)$ . We want to compute the derivative at  $t = 0$  of the map  $t \rightarrow \lambda_{1,\Omega_t}$ .

Shape analysis is the subject of an intense research activity. We refer for example to [16] for an introduction to this field. The shape derivative of the first eigenvalue  $\lambda_{p,\Omega}$  of the  $p$ -Laplacian,  $p > 1$ , has been computed in [13, 21]:

$$\frac{d}{dt} \lambda_{p,\Omega_t}|_{t=0} = -(p-1) \int_{\partial\Omega} \left| \frac{\partial u_p}{\partial \nu} \right|^p (V, \nu) dH^{n-1},$$

where  $u_p$  is the unique positive normalized eigenfunctions for  $\lambda_{p,\Omega}$  and  $\nu$  is the unit normal vector to  $\partial\Omega$ . What could be the shape derivative of the first eigenvalue of the 1-Laplacian is thus not obvious from this formula.

Our result is the following:

**Theorem.** *Let  $\Omega$  be a smooth bounded open subset of  $\mathbb{R}^n$ ,  $V : \mathbb{R}^n \rightarrow \mathbb{R}^n$  a smooth vector-field and  $\Omega_t = T_t(\Omega)$ , where the  $T_t$ 's are the diffeomorphisms defined by (5). Then*

$$\lambda_{1,\Omega_t} \rightarrow \lambda_{1,\Omega}$$

*as  $t \rightarrow 0$ . Moreover, if we assume that there exists a unique nonnegative eigenfunction  $u \in BV(\Omega)$  for  $\lambda_{1,\Omega}$  such that  $\int_{\Omega} |u| dx = 1$ , then  $u = |A|^{-1} \chi_A$  for some eigen-set  $A \subset \bar{\Omega}$ , and the map  $t \rightarrow \lambda_{1,\Omega_t}$  is differentiable at  $t = 0$  with*

$$\frac{d}{dt} \lambda_{1,\Omega_t}|_{t=0} = \int_{\partial^* A} (\operatorname{div} V - (\nu, DV\nu) + \lambda_{1,\Omega}(V, \nu)) \frac{dH^{n-1}}{|A|}, \quad (6)$$

*where  $\nu$  is the Radon-Nykodym derivative of  $\nabla \chi_A$  with respect to  $|\nabla \chi_A|$  (i.e.  $\nabla \chi_A = \nu |\nabla \chi_A|$  as measures), and  $\partial^* A$  denotes the reduced boundary of  $A$ , i.e. the part of the boundary of  $A$  at which can be defined a notion of unit normal vector in a measure theoretic sense (see e.g. [2, 11, 14, 26]).*

Let us assume that  $\Omega \subset \mathbb{R}^n$  is strictly convex. We then know from [4, 25] that there exists a unique Cheeger set  $A \subset \bar{\Omega}$ . Since  $A$  has minimum perimeter among all the subsets of  $\bar{\Omega}$  of finite perimeter and of volume  $|A|$ , it follows from [25] that  $\partial A$  is  $C^{1,1}$ . Hence the unit exterior normal vector to  $\partial A$  is defined  $H^{n-1}$  a.e. and Lipschitz. Its components are thus differentiable at  $H^{n-1}$  almost every point of  $\partial A$ . Note that this vector coincides with  $-\nu$   $H^{n-1}$ -a.e.. The mean curvature  $H_{\partial A}$  of  $\partial A$  is defined by  $H_{\partial A} = -\operatorname{div}_{\partial A} \nu$ , where  $\operatorname{div}_{\partial A}$  denotes the tangential derivative on  $\partial A$ . We can now write that

$$\operatorname{div} V - (\nu, DV\nu) = \operatorname{div}_{\partial A} V = \operatorname{div}_g V_{\partial A} - H_{\partial A}(V, \nu),$$

where  $V_{\partial A}$  denotes the tangential part of  $V$ , and  $\operatorname{div}_g$  the divergence operator of the manifold  $(\partial A, g)$ ,  $g$  being the metric on  $\partial A$  induced by the Euclidean metric (see e.g. [16]). We can thus rewrite (6) as

$$\frac{d}{dt} \lambda_{1,\Omega_t}|_{t=0} = \int_{\partial A} (\operatorname{div}_g V_{\partial A} - H_{\partial A}(V, \nu) + \lambda_{1,\Omega}(V, \nu)) \frac{dH^{n-1}}{|A|}$$

and thus

$$\frac{d}{dt} \lambda_{1,\Omega_t}|_{t=0} = \int_{\partial A} (\lambda_{1,\Omega} - H_{\partial A})(V, \nu) \frac{dH^{n-1}}{|A|}. \quad (7)$$

When  $A = \bar{\Omega}$ , a situation that happens when  $\Omega \subset \mathbb{R}^n$  is smooth convex and its curvature is less than  $|\partial \Omega|/((n-1)|\Omega|)$  (see [19] when  $n = 2$ , and [3] for an arbitrary  $n$ ), formula (7) writes as

$$\frac{d}{dt} \lambda_{1,\Omega_t}|_{t=0} = \int_{\partial \Omega} (\lambda_{1,\Omega} - H_{\partial \Omega})(V, \nu) \frac{dH^{n-1}}{|\Omega|},$$

where  $\nu$  is the inner unit normal to  $\partial \Omega$ . In the particular case when  $\Omega$  is a ball,  $A = \bar{\Omega}$  and  $H_{\partial \Omega}$  is constant, so that if we consider a deformation that preserves the volume, i.e. a vector-field  $V$  such that  $\operatorname{div} V = 0$ , we get

$$\frac{d}{dt} \lambda_{1,\Omega_t}|_{t=0} = 0.$$

Hence a ball is critical for such deformations.

The following section is devoted to the proof of the theorem.

## PROOF OF THE THEOREM

To simplify the notations, we let  $\lambda = \lambda_{1,\Omega}$  and  $\lambda_t = \lambda_{1,\Omega_t}$ .

Using the change of variable formula for functions of bounded variations [14], we can rewrite  $\lambda_t$  as

$$\lambda_t = \inf_{v \in BV(\Omega)} \frac{\int_{\bar{\Omega}} |D(x,t) \cdot \nu_v| C(x,t) |\nabla \bar{v}|}{\int_{\Omega} C(x,t) |v| dx}, \quad (8)$$

where  $D(x,t) = (DT_t(x))^{-1}$ ,  $C(x,t) = |\det(DT_t(x))|$ , and  $\nu_v$  is the Radon-Nikodym derivative of  $\nabla v$  with respect to  $|\nabla v|$ . Recall that  $\bar{v}$  denotes the extension of  $v$  to  $\mathbb{R}^n$  by 0 - see (2). As  $|\nu_v| = 1$   $|\nabla v|$  - a.e.,

$$\lambda_t \leq \inf_{v \in BV(\Omega)} \frac{\int_{\bar{\Omega}} |D(x,t) C(x,t) |\nabla \bar{v}|}{\int_{\Omega} C(x,t) |v| dx}. \quad (9)$$

Since  $|D(x,t)|, C(x,t) \rightarrow 1$  as  $t \rightarrow 0$  uniformly in  $x \in \bar{\Omega}$ , we deduce from (9) that

$$\limsup_{t \rightarrow 0} \lambda_t \leq \lambda. \quad (10)$$

We let  $u_t \in BV(\Omega_t)$  be a nonnegative eigenfunction for  $\lambda_t$  normalized by  $\int_{\Omega_t} u_t dx = 1$ , and  $v_t = u_t \circ T_t \in BV(\Omega)$ . Then  $(\bar{v}_t)$  is bounded in  $BV(\mathbb{R}^n)$ . Indeed if we denote by  $\bar{\nu}_t$  the Radon-Nikodym derivative of  $\nabla u_t$  with respect to  $|\nabla u_t|$ , we have

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla \bar{v}_t| &= \int_{\bar{\Omega}_t} |(DT_t^{-1})^{-1} \bar{\nu}_t| |\det DT_t^{-1}| |\nabla \bar{u}_t| \leq (1 + o(1)) \int_{\bar{\Omega}_t} |\nabla \bar{u}_t| \\ &= (1 + o(1)) \lambda_t, \end{aligned} \quad (11)$$

and

$$\int_{\Omega} v_t dx = \int_{\Omega_t} u_t |\det DT_t^{-1}| dx = 1 + o(1). \quad (12)$$

We can thus assume that the  $v_t$ 's converge to some nonnegative  $v \in L^1(D)$  in  $L^1(D)$  and a.e. in  $D$ , where  $D$  is a smooth bounded open subset of  $\mathbb{R}^n$  containing both  $\Omega$  and the  $\Omega_t$ 's. We denote by  $u$  the restriction of  $v$  to  $\Omega$ . Then  $u \in BV(\Omega)$ ,  $u \geq 0$ , and in view of (12),

$$\int_{\Omega} u dx = 1.$$

The lower semi-continuity of the total variation and (11) then give

$$\lambda \leq \int_{\bar{\Omega}} |\nabla \bar{u}| \leq \int_{\mathbb{R}^n} |\nabla v| \leq \liminf_{t \rightarrow 0} \int_{\mathbb{R}^n} |\nabla \bar{v}_t| \leq \liminf_{t \rightarrow 0} \lambda_t.$$

We then deduce with (10) that

$$\lambda_t \rightarrow \lambda = \int_{\bar{\Omega}} |\nabla \bar{u}|,$$

as  $t \rightarrow 0$ , and then that

$$\int_{\bar{\Omega}} |\nabla \bar{v}_t| \rightarrow \lambda = \int_{\bar{\Omega}} |\nabla \bar{u}|, \quad (13)$$

as  $t \rightarrow 0$ . This proves the first part of the theorem.

Let us note for a future use that, as in [17], it follow from (13) that  $|\nabla \bar{v}_t| \rightarrow |\nabla \bar{u}|$  weakly in the sense that

$$\int_{\mathbb{R}^n} \phi |\nabla \bar{v}_t| \rightarrow \int_{\mathbb{R}^n} \phi |\nabla \bar{u}| \quad (14)$$

for any  $\phi \in C(\mathbb{R}^n)$  with compact support.

We now prove the differentiability of the map  $t \rightarrow \lambda_t$  and the formula (6). We use  $u$  as a test-function in (8) to estimate  $\lambda_t$ , so that

$$\lambda_t - \lambda \leq \frac{\int_{\bar{\Omega}} |D(x, t) \nu| C(x, t) |\nabla \bar{u}|}{\int_{\Omega} C(x, t) u \, dx} - \lambda, \quad (15)$$

where  $\nu$  is the Radon-Nikodym derivative of  $\nabla u$  with respect to  $|\nabla u|$ . Since  $|\nu| = 1$   $|\nabla u|$ -a.e., we can assume that  $|\nu| = 1$  everywhere. Direct computations give that

$$C(x, t) = \det(DT_t(x)) = 1 + t \operatorname{div} V(x) + o(t), \quad (16)$$

and

$$|D(x, t) \nu| = 1 - (\nu, DV(x) \nu) t + o(t), \quad (17)$$

where the  $o(t)$  is uniform in  $x$ . Thus (15) becomes

$$\begin{aligned} \lambda_t - \lambda &\leq \frac{\lambda + t \int_{\bar{\Omega}} (\operatorname{div} V - (\nu, DV \nu)) |\nabla \bar{u}| + o(t)}{1 + t \int_{\Omega} u \operatorname{div} V \, dx + o(t)} - \lambda \\ &= t \left( \int_{\bar{\Omega}} \operatorname{div} V (|\nabla \bar{u}| - \lambda u \, dx) - \int_{\bar{\Omega}} (\nu, DV \nu) |\nabla \bar{u}| \right) + o(t). \end{aligned}$$

Hence

$$\limsup_{t \rightarrow 0^+} \frac{\lambda_t - \lambda}{t} \leq \int_{\bar{\Omega}} \{(\operatorname{div} V - (\nu, DV \nu)) |\nabla \bar{u}| - \lambda u \operatorname{div} V \, dx\}, \quad (18)$$

and

$$\liminf_{t \rightarrow 0^-} \frac{\lambda_t - \lambda}{t} \geq \int_{\bar{\Omega}} \{(\operatorname{div} V - (\nu, DV \nu)) |\nabla \bar{u}| - \lambda u \operatorname{div} V \, dx\}. \quad (19)$$

It remains to prove the opposite inequalities. We use  $\bar{v}_t$  as a test-function to estimate  $\lambda$ , so that

$$\lambda_t - \lambda = \int_{\Omega_t} |\nabla \bar{u}_t| - \lambda \geq \int_{\bar{\Omega}} |D(x, t) \nu_t| C(x, t) |\nabla \bar{v}_t| - \frac{\int_{\bar{\Omega}} |\nabla \bar{v}_t|}{\int_{\Omega} v_t \, dx},$$

where  $\nu_t$  denotes the Radon-Nykodym derivative of  $\nabla \bar{v}_t$  with respect to  $|\nabla \bar{v}_t|$ . As previously, we can assume that  $|\nu_t| = 1$  everywhere. In view of (16) and (17), we obtain

$$\lambda_t - \lambda \geq \int_{\bar{\Omega}} |\nabla \bar{v}_t| + t \int_{\bar{\Omega}} (\operatorname{div} V - (\nu_t, DV \nu_t)) |\nabla \bar{v}_t| - \frac{\int_{\bar{\Omega}} |\nabla \bar{v}_t|}{\int_{\Omega} v_t \, dx} + o(t). \quad (20)$$

Since  $\operatorname{div} V \in C(\bar{\Omega})$ , we get thanks to (14) that

$$\int_{\bar{\Omega}} \operatorname{div} V |\nabla \bar{v}_t| \rightarrow \int_{\bar{\Omega}} \operatorname{div} V |\nabla \bar{u}|.$$

Independently,

$$\int_{\Omega} v_t dx = \int_{\Omega} u_t |\det DT_t^{-1}| dx$$

with

$$\det DT_t^{-1} = 1 - t \operatorname{div} V + o(t),$$

so that

$$\int_{\Omega} v_t dx = 1 - t \int_{\Omega_t} u_t \operatorname{div} V dx + o(t) = 1 - t \int_{\Omega} u \operatorname{div} V dx + o(t).$$

Thus

$$\begin{aligned} \frac{\int_{\bar{\Omega}} |\nabla \bar{v}_t|}{\int_{\Omega} v_t dx} &= \int_{\bar{\Omega}} |\nabla \bar{v}_t| + t \int_{\bar{\Omega}} |\nabla \bar{v}_t| \int_{\Omega} u \operatorname{div} V dx + o(t) \\ &= \int_{\bar{\Omega}} |\nabla \bar{v}_t| + t \lambda \int_{\Omega} u \operatorname{div} V dx + o(t), \end{aligned}$$

where the last equality follows from (13). Hence (20) becomes

$$\lambda_t - \lambda \geq t \int_{\bar{\Omega}} \operatorname{div} V (|\nabla \bar{u}| - \lambda u) dx - t \int_{\bar{\Omega}} (\nu_t, DV \nu_t) |\nabla \bar{v}_t| + o(t).$$

Eventually, in view of (13) and the weak convergence of  $\nabla \bar{v}_t$  to  $\nabla \bar{u}$ , which follows from the  $L^1$  convergence of the  $\bar{v}_t$  to  $\bar{u}$ , we can apply Reshetnyak' theorem [2, 23, 22] to get that

$$\int_{\bar{\Omega}} g(x, \nu_t(x)) |\nabla \bar{v}_t| \rightarrow \int_{\bar{\Omega}} g(x, \nu(x)) |\nabla \bar{u}|$$

for any continuous function  $g : \bar{\Omega} \times S \rightarrow \mathbb{R}$ , where  $S$  denotes the unit sphere of  $\mathbb{R}^n$ . In particular,

$$\int_{\bar{\Omega}} (\nu_t, DV \nu_t) |\nabla \bar{v}_t| = \int_{\bar{\Omega}} (\nu, DV \nu) |\nabla \bar{u}| + o(1).$$

Hence

$$\lambda_t - \lambda \geq t \int_{\bar{\Omega}} \{(\operatorname{div} V - (\nu, DV \nu)) |\nabla \bar{u}| - \lambda u \operatorname{div} V dx\} + o(t),$$

and thus

$$\limsup_{t \rightarrow 0^+} \frac{\lambda_t - \lambda}{t} \geq \int_{\bar{\Omega}} \{(\operatorname{div} V - (\nu, DV \nu)) |\nabla \bar{u}| - \lambda u \operatorname{div} V dx\},$$

and

$$\liminf_{t \rightarrow 0^-} \frac{\lambda_t - \lambda}{t} \leq \int_{\bar{\Omega}} \{(\operatorname{div} V - (\nu, DV \nu)) |\nabla \bar{u}| - \lambda u \operatorname{div} V dx\}.$$

Since by assumption  $u$  is the unique normalized eigenfunctions for  $\lambda$ , we deduce from these two inequalities and (18), (19) that the map  $t \rightarrow \lambda_t$  is differentiable at  $t = 0$  with

$$\frac{d}{dt} \lambda_t|_{t=0} = \int_{\bar{\Omega}} \{(\operatorname{div} V - (\nu, DV \nu)) |\nabla \bar{u}| - \lambda u \operatorname{div} V dx\}.$$

Eventually, as there always exists an extremal for the problem (4), there exists a set of finite perimeter  $A \subset \bar{\Omega}$  such that  $u = |A|^{-1} \chi_A$ . It then follows from geometric measure theory that  $|\nabla \bar{u}| = |A|^{-1} H_{\partial^* A}^{n-1}$  (see e.g. [2, 11, 14, 26]). The previous formula becomes

$$\frac{d}{dt} \lambda_{t|t=0} = \int_{\partial^* A} (\operatorname{div} V - (\nu, DV\nu)) \frac{dH^{n-1}}{|A|} - \lambda \int_A \operatorname{div} V \frac{dx}{|A|},$$

from which we deduce (6) using Green' formula for sets of finite perimeter. This ends the proof of the theorem.

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## REFERENCES

- [1] F.J. Almgren, Existence and regularity almost everywhere of solutions to elliptic variational problems, *Mem. Am. Math. Soc.*, 165, vol.4, 1976.
- [2] L. Ambrosio, N. Fusco, D. Pallara, Functions of bounded variations and free discontinuity problems, *Oxford Mathematical Monographs*, The Clarendon Press, Oxford University Press, New-York, 2000.
- [3] F. Alter, V. Caselles, A. Chambolle, A characterization of convex calibrable sets in  $R^n$ , *Math. Ann.*, 332 (2), 2005, 329-366.
- [4] V. Caselles, A. Chambolle, M. Novaga, Uniqueness of the Cheeger set of a convex body, *preprint*.
- [5] I. Chavel, *Riemannian geometry – a modern introduction*, Cambridge Tracts in Mathematics, 108, Cambridge University Press, Cambridge, 1993.
- [6] J. Cheeger, A lower bound for the smallest eigenvalue of the Laplacian, in *Problems in Analysis, A Symposium in honor of S. Bochner*, Princeton Univ. Press, 1970, 195-199.
- [7] F. Demengel, On some nonlinear equation involving the 1-Laplacian and trace map inequalities, *Nonlinear Analysis*, 48, 2002, 1151-1163.
- [8] F. Demengel, On some nonlinear partial differential equations involving the 1-Laplacian and critical Sobolev exponent, *ESAIM*, 4, 1999, 667-686.
- [9] F. Demengel, F. De Vuyst, M. Motron, A numerical approach of the first eigenvalue for the 1-Laplacian on the square and other particular sets, *Preprint*, 2002.
- [10] E. De Giorgi, Frontiere orientate di misura minima, Seminario di Matematica della Scuola Normale Superiore di Pisa, Editrice Tecnico Scientifica, Pisa, 1961.
- [11] L.C. Evans, R.F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Math., CRC Press, Ann Harbor, 1992.
- [12] V. Fridman, B. Kawohl, Isoperimetric estimates for the first eigenvalue of the  $p$ -Laplace operator and the Cheeger constant, *Comment. Math. Univ. Carolinae*, 44, 2003, 659-667.
- [13] J. Garcia Melian, J. Sabina De Lis, On the perurbation of eigenvalues for the  $p$ -Laplacian, *C.R. Acad. Sci. Paris, Série 1*, 332, 2001, 893-898.
- [14] Giusti, E., Minimal surfaces and functions of bounded variation, *Monographs in Mathematics*, Birkhäuser, 1984.
- [15] E. Gonzalez, U. Massari, I. Tamanini, On the regularity of boundaries of sets minimizing perimeter with a volume constraint, *Indiana Univ. Math. J.*, 32, 1983, 25-37.
- [16] A. Henrot, M. Pierre, Variation et optimisation de formes - une analyse gomtrique, *Mathematiques et applications* 48, Berlin, New York, Springer, 2005.
- [17] E. Hebey, N. Saintier, Stability and perturbations of the domain for the first eigenvalue of the 1-Laplacian, *Archiv der Mathematik*, 86, (4), 2006, 340-351.
- [18] I. R. Ionescu, T. Lachand-Robert, Generalized Cheeger sets related to landslides, *Calc. Var. and PDE's*, 23 (2005), 227-249.
- [19] B. Kawohl, T. Lachand-Robert, Characterization of Cheeger sets for convex subsets of the plane, *Pacific J. Math.*, 225 (1), 2006, 103-118.

- [20] B. Kawohl, F. Schuricht, Dirichlet problems for the 1-Laplace operator, including the eigenvalue problem, *Comm. Contemp. Math.*, to appear.
- [21] P.D. Lamberti, A differentiability result for the first eigenvalue of the  $p$ -Laplacian upon domain perturbation, *Nonlinear analysis and applications: to V. Lakshmikantham on his 80th birthday*, vol. 1, 2, Kluwer Acad. Publ., Dordrecht, 2003, 741-754.
- [22] S. Luckhaus, L. Modica, The Gibbs-Thompson relation within the gradient theory of phase transitions, *ARMA*, 107 (1), 71-83.
- [23] Yu. G. Reshetnyak, Weak convergence of completely additive vector functions on a set, *Siberian Math. J.*, 9, 1968, 1039-1045; translated from *Sibirskii Matematicheskii Zhurnal*, 9, 1968, 1386-1394.
- [24] N. Saintier, Estimates of the best Sobolev constant of the embedding of  $BV(\Omega)$  into  $L^1(\partial\Omega)$  and related shape optimization problems, *submitted*.
- [25] E. Stredulinsky, W.P. Ziemer, Area minimizing sets subject to a volume constraint in a convex set, *J. Geom. Anal.*, 7, 1997, 653-677.
- [26] W.P. Ziemer, Weakly differentiable functions. Sobolev spaces and functions of bounded variations, *Graduate Texts in Mathematics* 120, Springer-Verlag, 1989.

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